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Non-dimensional numbers as ratios of characteristic times

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INTRODUCTION

FOR SCORES of years we have gained physical insight into the field of fluid mechanics, heat and mass transfer, by casting non-dimensional numbers as ratios of characteristic forces, characteristic energies, or characteristic times and lengths. Such interpretations are both simple and appropriate if we deal with problems where two characteristic terms are present in one of the conservation equations. For instance, in the pure hydrodynamic case when only the inertia and the viscosity terms appear, the Reynolds number admits the simple and appropriate interpretation of being the ratio of the typical inertia to viscous force. Equally well, comparison of the convected with the conducted energies in the energy equation leads to the definition of the Peclet number. On the other hand, in the case of forced convection, the Prandtl number is introduced through the combined use of both momentum and energy equations, and it is thus not possible to interpret it in terms of either energy or force ratios; it is commonly interpreted as the square root ratio of the momentum and thermal boundary layer thicknesses. However, this interpretation is not universal and fails in the case of natural convection for low Prandtl numbers where the two boundary layers coincide. To illustrate further difficulties of interpretation, consider the question of the physical meaning of the Grashof number. Textbooks and research papers alike often give the interpretation that it is the ratio of the buoyant to viscous forces (for the dangers of such a physical meaning, see the book by Bejan [1]). This is not satisfactory since the Grashof number does not involve a velocity to define the viscous term. If one introduces artificially a characteristic unspecified velocity, then one can correctly state that Gr is the ratio of the product of the inertia and buoyancy forces divided by the square of the viscous forces. But then what is the physical significance of the Rayleigh number that correctly describes rates of heat transfer? And what about the natural convection case in the limit of low Prandtl numbers when the so-called Boussinesq rather than the Rayleigh number is the pertinent one? To evoke the physical interpretation of the Grashof and Prandtl numbers as given above, and say that $Ra = Gr \cdot Pr$ or that $Bo = Gr \cdot Pr^2$ is not satisfactory if one wishes to create a physical meaning for these quantities to promote direct understanding.

To make the inconsistency in the interpretation of non-dimensional numbers even more pointed, consider the situation in which yet another body force is introduced, such as the ponderomotive force $\mathbf{J} \times \mathbf{B}$ in the case of magneto-fluid mechanics. To be more specific consider the non-dimensional combination $Ly = M^2/Gr^{1/2}$, which arises in the problem of natural convection in the presence of a magnetic field. If one adopts the conventional physical definition for the square of the Hartmann number M (the ratio of ponderomotive to viscous force), and the one for the Grashof number as given above through the artificial introduction of an unspecified velocity, then the interpretation for Ly is the one given in the *Handbook of Chemistry and Physics* [2], namely the ratio of the square of the ponderomotive force divided by the product of the buoyant and the inertia forces. The weakness of this interpretation lies in the fact that there is no characteristic velocity involved in the Ly grouping that could identify the inertia forces, but what is worse, viscosity is absent, which could incorrectly be implied by its presence in the Grashof and Hartmann numbers. Finally, as a further example consider a problem involving a phase change such as nucleate boiling in the presence of a magnetic field. A simple analysis by Lykoudis [3] has shown that the heat transfer coefficient depends on a new dimensionless number that was difficult to associate with a simple and unambiguous physical meaning, since it emerged as a combination of thermodynamic variables, superheats, transport properties and of course the intensity of the magnetic field.

From the examples cited above it appears that better and more direct physical interpretations for non-dimensional numbers are desirable. It is proposed in this paper to assign to all non-dimensional numbers, as they originate from the conservation equations of momentum, heat, mass transfer, and Maxwell's equations, a physical meaning through characteristic times such as the ones associated with the processes of diffusion (momentum, heat, mass, electricity), time measured by a clock for a particle to travel a given characteristic distance, or the 'free fall' time for a particle to fall freely in an atmosphere of a given density, and additional times as they will be developed later in the text. It is not that characteristic times have not been used by researchers in the past, there are plenty of examples such as, for instance, Bejan [1] (in particular pp. 163 and 213), but this point of view has been used sporadically and not carried through in the systematic way proposed in the present paper. When one does so, it will be seen that all non-dimensional groupings admit simple physical meanings for all geometries and flow conditions having the advantage of directness and universality.

DEFINITIONS AND DERIVATIONS

We first make a list of the order of magnitude of some terms that appear in the equations of conservation.

$$\rho U/t \propto \text{non-steady inertia force per unit volume} \quad (1)$$

$$\rho U^2/L \propto \text{convective inertia force per unit volume} \quad (2)$$

$$\Delta p/L \propto \text{pressure forces per unit volume} \quad (3)$$

$$\mu U/L^2 \propto \text{viscous force per unit volume} \quad (4)$$

$$g\Delta\rho \propto \text{buoyancy force per unit volume} \quad (5)$$

$$\sigma/L^2 \propto \text{surface tension force per unit volume} \quad (6)$$

$$\sigma_e U B^2 \propto \text{ponderomotive force per unit volume} \quad (7)$$

$$\rho c_p U \Delta T/L \propto \text{thermal convection per unit volume} \quad (8)$$

$$k\Delta T/L^2 \propto \text{thermal conduction per unit volume} \quad (9)$$

$$U\Delta\rho/L \propto \text{mass convected per unit volume} \quad (10)$$

$$D_m \Delta\rho/L^2 \propto \text{mass diffused per unit volume.} \quad (11)$$

Now equate (1) and (2) to define the convective time t_c

$$t_c \propto L/U. \quad (12)$$

Equate (1) and (5) along with $U \propto L/t$ to define the 'free fall' time t_f

$$t_f \propto [L/g(\Delta\rho/\rho)]^{1/2}. \quad (13)$$

Equate (2) and (3) to find the characteristic time t_p needed to empty a vessel of size L under a constant pressure difference Δp

$$t_p \propto L(\Delta p/\rho)^{-1/2}. \quad (14)$$

Equate (1) with (6) to define the characteristic surface tension time (the time it takes for the inertia forces to overcome the surface tension forces)

$$t_{st} \propto (\rho L^3/\sigma)^{1/2}. \quad (15)$$

We classify the times t_c, t_f, t_p and t_{st} as 'non-dissipative times'. It is clear that the words 'dissipative' and 'non-dissipative' constitute an abuse of language, but it helps here to show their physical origin.

Equate (1) with (4), and using (1), equate (8) with (9) and (10) with (11), to define the corresponding diffusion characteristic times for momentum, heat and mass

$$t_M \propto L^2/\nu \quad (16)$$

$$t_H \propto L^2/\alpha \quad (17)$$

$$t_m \propto L^2/D_m. \quad (18)$$

Use now the ratio of the heat capacity per unit volume with the heat of vaporization per unit volume to define the Jakob number:

$$Ja = \text{Jakob number} \propto \rho c_p \Delta T/\rho_v \lambda. \quad (19)$$

In terms of the Jakob number, modify the heat diffusion time to define the 'Jakob time' which is the time needed for an evaporation front to move a distance L. (To my knowledge this is a new definition, but a rather useful one for two-phase heat transfer problems, as for instance, is the case of bubble growth in a superheated liquid.)

$$t_{Ja} \propto (L^2/\alpha)/(Ja)^2 \propto t_H/(Ja)^2 \quad (20)$$

Let us now consider magneto-fluid mechanical problems.

Equate (1) with (7) to define the so-called 'roll-over' time, to indicate the time it takes for a small velocity fluctuation to be damped out by the magnetic field B for an electrically conducting fluid:

$$t_B \propto \rho/\sigma_e B^2. \quad (21)$$

From a combination of Ampere's, Faraday's, and Ohm's laws, equate the Laplacian of the magnetic field B with the diffusion of electric charge through the medium's electrical conductivity σ_e to find the characteristic time t_e it takes for B to diffuse the thickness L:

$$T_e \propto L^2/(1/\mu_e \cdot \sigma_e). \quad (22)$$

The times t_M, t_H, t_m, t_B, t_e and t_{Ja} originate from dissipative processes. We can now recall the definitions of non-dimensional numbers as they emerge in fluid mechanics, heat and mass transfer, and show that they can all be expressed in terms of the above characteristic times. We have:

$$Re = \text{Reynolds number} = UL/\nu \propto t_M/t_c \quad (23)$$

$$Pe = \text{Peclet number} = UL/\alpha \propto t_H/t_c \quad (24)$$

$$Le = \text{Lewis number} = UL/D_m \propto t_m/t_c \quad (25)$$

$$(Re)_m = \text{magnetic Reynolds number} = \sigma_e \mu_e LU \propto t_e/t_c \quad (26)$$

$$Fr = \text{Froude number} = U \cdot [Lg(\Delta\rho/\rho)]^{-1/2} \propto t_f/t_c. \quad (27)$$

Note here that in the regular definition of the Fourde number we have $\Delta\rho/\rho = 1$.

$$We = \text{Weber number} = \rho U^2 L/\sigma \propto t_{st}/t_c \quad (28)$$

$$Mo = \text{Morton number} = g\mu^4/\rho\sigma^3 \propto t_{st}^3 \cdot t_f/t_M^4 \quad (29)$$

$$Pr = \text{Prandtl number} = \nu/\alpha \propto t_H/t_M \quad (30)$$

$$Sc = \text{Schmidt number} = \nu/D_m \propto t_m/t_M \quad (31)$$

$$Pr_m = \text{magnetic Prandtl number} = \sigma_e \mu_e \nu \propto t_e/t_M \quad (32)$$

$$Gr = \text{Grashof number} = g(\Delta\rho/\rho)L^3/\nu^2 \propto (t_M/t_f)^2 \quad (33)$$

$$Bo = \text{Boussinesq number} = g(\Delta\rho/\rho)L^3/\alpha^2 \propto (t_H/t_f)^2 \quad (34)$$

$$Ra = \text{Rayleigh number} = g(\Delta\rho/\rho)L^3/\alpha\nu \propto (t_M \cdot t_H)/t_f^2 \quad (35)$$

$$Ma = \text{Marangoni number} = (\partial\sigma/\partial T)(\Delta T)L/\mu\alpha \propto (t_M \cdot t_H)/t_{st}^2 \quad (36)$$

$$Eo = \text{Eotvos or Bond number} = [\rho g L^2/\sigma]^{1/2} \propto t_{st}/t_f \quad (37)$$

$$M = \text{Hartmann number} = BL(\sigma_e/\mu)^{1/2} \propto (t_M/t_B)^{1/2} \quad (38)$$

$$N = \text{magnetic interaction parameter} = \sigma_e B^2 L/\rho U \propto t_c/t_B \quad (39)$$

$$Ly = \text{Lykoudis number} = \sigma_e B^2[g(\Delta\rho/\rho)/L]^{-1/2}\rho \propto t_f/t_B \quad (40)$$

$$M/Re = B(\sigma_e\mu)^{1/2}/\rho U \propto t_c/(t_M \cdot t_B)^{1/2}. \quad (41)$$

The non-dimensional number Λ that was introduced in ref. [2] has the following meaning:

$$\Lambda = \text{magnetic interaction boiling number} = B^2 \Delta T(\sigma_e \alpha c_p^2 \rho_l^2 T_{sat}/\rho_v^3 \lambda^3) \propto t_p^2/t_{Ja} t_B. \quad (42)$$

One could proceed with many more dimensionless numbers, some of which exist in the literature and others that do not, such as for instance the Mach number, introduced in problems where compressibility is important, or numbers involving the magnetic pressure as used in plasma physics. However, the above collection should suffice to illustrate the merits of the suggested definitions.

DISCUSSION

Let us now look at the physical meaning of the non-dimensional numbers through the characteristic times as developed above. The Reynolds number appears now as the competition between the time t_c it takes a particle to travel a physical distance around an object, and the time it takes for vorticity to diffuse over the same distance. Analogous interpretation can be seen for the Peclet, Lewis and magnetic Reynolds numbers as the diffusion of heat, mass and electric charge are compared with the time t_c. The Prandtl number is the ratio of the two characteristic times for momentum and heat diffusion. The Grashof number is simply an expression of the relative characteristic times for the diffusion of momentum and the free fall time. It is clearly an appropriate measure of when natural convection becomes turbulent (when $t_M > t_f$) and not a measure of heat transfer. Also, the answer to the question as to when forced convection dominates natural convection is the statement $t_f > t_c$, which is far more clear and direct compared to the familiar inequality $Re^2 \gg Gr$. For heat transfer, on the other hand, the Rayleigh number does not need to be understood via the Grashof and Prandtl numbers. It involves the competition of the harmonic mean between the two diffusion times of heat and momentum, and the free fall time. The Boussinesq time involves only the characteristic times of diffusion of heat and free fall as it should, since for the case of media of high thermal conductivity the diffusion time for momentum transfer is very long; its meaning is clear and unambiguous compared to trying to detect it from the regular definition of

$Bo = Gr \cdot Pr^2$. Note also that the Marangoni number is a close cousin of the Rayleigh number as the diffusive combination $(t_M \cdot t_H)^{1/2}$ is compared with the time t_{st} for the Marangoni number, and t_f for the Rayleigh number.

Going to two-phase problems we can immediately write, for instance, the criterion for the presence of skirts in bubbles as $t_{st} > t_M$ rather than the equivalent one of $We > Re$. Also, more physical insight can be gained by analyzing the combination $Re \cdot Mo^{1/4}$ that appears in inequalities which determine the shape of bubbles and bubble drag coefficients, by writing out its equivalent form in terms of characteristic times as:

$$Re \cdot Mo^{1/4} \propto [t_{st}^{3/4} \cdot t_f^{1/4}/t_c].$$

One can see that in this combination the momentum diffusion time, prominent in the Morton number, is replaced with the convective time t_c.

Let us now take some examples from magneto-fluid mechanics: the Hartmann and magnetic interaction parameters are measures of the momentum diffusion and the convective times as they compete with the magnetic roll-over time. On the other hand, the combination M/Re, which is a measure of the skin friction in laminar MHD duct flow, emerges as the ratio of the harmonic mean between the two dissipative times t_M and t_B with the convective time t_c. The Ly grouping which correlates ratios of heat transfer rates in natural convection with and without magnetic fields, has the simple meaning of the competition between the characteristic free fall time and the roll-over time. Again, this is a straightforward physical interpretation which is masked by the current definition (correct in strict algebraic terms) involving the Grashof and Hartmann numbers. Finally, note that the magnetic interaction boiling parameter Λ turns out to be the ratio of the non-diffusive time t_p and the harmonic mean of the dissipative times t_{Ja} and t_B.

It is worth observing that in all the definitions of the non-dimensional numbers given here the dissipative and non-dissipative times never seem to mix, but always appear as competitors. For instance, in all of the cases cited, whenever more than one diffusion mechanism is involved, the ratio of the characteristic times appears as a ratio of a non-dissipative time, and the product of two diffusion, dissipative times. The explanation can be sought by observing that the so-called 'dissipative' times are involved with second derivatives, whereas the so-called 'non-dissipative' times are involved with first order derivatives.

From the discussion above it is clear that by using characteristic times to define non-dimensional numbers one avoids the difficulty of having to mix quantities of energy, force, mass, etc. for their interpretation, a procedure that can lead to confusion and often times to erroneous assessment. With the help of the conservation and Maxwell's equations, characteristic times can easily be defined that provide a common, consistent measure of comparison for all non-dimensional parameters no matter what their origin is.

Finally, it needs to be stated that nothing prevents the attachment to non-dimensional numbers of other special *ad hoc* meanings as they might be pertinent in problems involving various geometries and flow conditions. Here again, Bejan [1] gives a number of good examples of what can be done with the correct physical scaling in giving powerful order of magnitude answers. The definitions given here can stand parallel to them without confusion.

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